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FIELDS, PARTICLES, AND NUCLEI

Quark and Gluon Condensates at a Finite Isospin Chemical Potential

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The nonperturbative vacuum of quantum chromodynamics at a finite isospin chemical potential has been studied. Low-energy relations for the quark and gluon condensates have been derived ab initio. Analytical expressions for the quark and gluon condensates in the pion-condensate phase have been obtained at the tree level of the chiral perturbation theory. It has been shown that the quark condensate decreases with increasing μ_I, whereas the gluon condensate increases.

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1. Pion condensation in nuclear matter (at finite baryon density) has long been studied [1–5]. At the beginning of the 2000s, it was shown [6–9] that the condensation of charged pions at zero baryon density and temperature occurs at the critical isospin chemical potential $\mu_I^c = m_\pi$. This phenomenon was studied in numerous works within the chiral perturbation theory, Nambu–Jona-Lasinio model, quark–meson model, and their extensions by the inclusion of a Polyakov loop (see [10, 11] and recent review [12]). It was assumed that a new type of compact stars—pion stars—based on this phenomenon can exist [10, 13]. Condensates in quantum chromodynamics (QCD) in isospin matter were studied in [14].

Relations following from the symmetry properties of the theory play an important role in quantum field theories. Searches for symmetries and constraints imposed by these symmetries on the physical characteristics of the system are of particular importance in the QCD theory with confinement, where composite states, hadrons, are "observables." Low-energy theorems or Ward identities (scale and chiral) are fundamentally important to understand nonperturbative vacuum properties of QCD. Low-energy theorems of QCD were derived at the beginning of the 1980s [15–17]. Low-energy theorems of QCD, which follow from the general symmetry properties and are independent of the details of the confinement mechanism, allow obtaining information that sometimes cannot be obtained by any other methods. In addition, they can be used "physically reasonable" constraints when developing effective theories and various models of QCD vacuum. Low-energy theorems in QCD at $T \neq 0$ and $\mu_q \neq 0$ were derived in [18, 19]. Low-energy theorems in the presence of a magnetic field and their applications to various physical processes were analyzed in [20–30]. Low-energy relations for the energy–momentum tensor at finite temperature were studied in [31–35].

In this work, the behavior of the quark and gluon condensates at a finite isospin chemical potential in the pion-condensate phase is studied.

2. In the Euclidean formulation, the partition function of QCD in the presence of an isospin chemical potential μ_I can be represented in the form

$$Z = \int [DA] \prod_{q=u,d} [D\bar{q}][Dq] \exp\left\{ -\int_{V_4} d^4x \mathcal{L} \right\}. \qquad (1)$$

Here,

$$\mathcal{L} = \frac{1}{4g_0^2}(G_{\mu\nu}^a)^2 + \sum_{q=u,d} \bar{q}\left[\gamma_\mu\left(\partial_\mu - i\frac{\lambda^a}{2}A_\mu^a\right) + \frac{1}{2}\mu_I\gamma_0\tau_3 + m_{0q}\right]q \qquad (2)$$

is the QCD Lagrangian involving quarks with flavor $q = (u, d)$ and the bare mass m_{0q}, whereas the ghost and gauge-fixing terms are not written explicitly for simplicity. The pressure is given by the expression $V_4 P(\mu_I, m_{0u}, m_{0d}) = \ln Z$. The relation for the gluon condensate $\langle G^2 \rangle \equiv \langle (G_{\mu\nu}^a)^2 \rangle$ follows from Eq. (1):

$$\langle G^2 \rangle(\mu_I, m_{0u}, m_{0d}) = -4\frac{\partial P}{\partial(1/g_0^2)}. \qquad (3)$$

The system described by the partition function given by Eq. (1) is characterized by the set of dimensional parameters M, μ_I, and $m_{0q}(M)$ and the dimensionless charge $g_0^2(M)$, where M is the ultraviolet-cut-off mass. At the same time, the renormalized pressure P_R can be considered and, with the use of dimensional and renormalization group properties, Eq. (3) can be represented in the form including the derivatives with respect to both the physical parameter μ_I and renormalized masses m_q.

The phenomenon of dimensional transmutation leads to the appearance of the nonperturbative dimensional parameter

$$\Lambda = M\exp\left\{\int_{\alpha_s(M)}^{\infty}\frac{d\alpha_s}{\beta(\alpha_s)}\right\}, \tag{4}$$

where $\alpha_s = g_0^2/4\pi$ and $\beta(\alpha_s) = d\alpha_s(M)/d\ln M$ is the Gell-Mann–Low function. It is well known that the quark mass has an anomalous dimension and depends on the scale M. The renormalization group equation for the running mass has the form $d\ln m_0/d\ln M = -\gamma_m$, and the $\overline{MS}$ scheme is used, where β and γ_m are independent of the quark mass [19, 36]. The renormalization group invariant mass has the form

$$m_q = m_{0q}(M)\exp\left\{\int^{\alpha_s(M)}\frac{\gamma_{m_q}(\alpha_s)}{\beta(\alpha_s)}d\alpha_s\right\}. \tag{5}$$

Since the pressure is a renormalization group invariant quantity, its anomalous dimension is zero. Therefore, P_R has only a normal (canonical) dimension equal to 4. In view of the renormalization group invariance of the quantity Λ, P_R can be written in the most general form

$$P_R = \Lambda^4 f\left(\frac{\mu_I}{\Lambda}, \frac{m_u}{\Lambda}, \frac{m_d}{\Lambda}\right), \tag{6}$$

where f is a certain function. It follows from Eqs. (4)–(6) that

$$\frac{\partial P_R}{\partial(1/g_0^2)} = \frac{\partial P_R}{\partial\Lambda}\frac{\partial\Lambda}{\partial(1/g_0^2)} + \sum_q\frac{\partial P_R}{\partial m_q}\frac{\partial m_q}{\partial(1/g_0^2)}, \tag{7}$$

$$\frac{\partial m_q}{\partial(1/g_0^2)} = -4\pi\alpha_s^2 m_q\frac{\gamma_{m_q}(\alpha_s)}{\beta(\alpha_s)}. \tag{8}$$

In view of (3), the gluon condensate is given by the expression

$$\begin{aligned}&\langle G^2\rangle(\mu_I, m_u, m_d)\\ &= -\frac{16\pi\alpha_s^2}{\beta(\alpha_s)}\left(4 - \mu_I\frac{\partial}{\partial\mu_I} - \sum_q(1+\gamma_{m_q})m_q\frac{\partial}{\partial m_q}\right)P_R.\end{aligned} \tag{9}$$

It is convenient to choose the scale large enough to take the lowest order in the expansion of the Gell-Mann–Low function $\beta(\alpha_s) \to -b\alpha_s^2/2\pi$, where $b = (11N_c - 2N_f)/3$ and $1+\gamma_m \to 1$. Thus, the equations for condensates have the form

$$\begin{aligned}&\langle G^2\rangle(\mu_I)\\ &= \frac{32\pi^2}{b}\left(4 - \mu_I\frac{\partial}{\partial\mu_I} - \sum_q m_q\frac{\partial}{\partial m_q}\right)P_R \equiv \hat{D}P_R,\end{aligned} \tag{10}$$

$$\langle\bar{q}q\rangle(\mu_I) = -\frac{\partial P_R}{\partial m_q}. \tag{11}$$

3. In QCD, the condensates $\langle\bar{q}q\rangle(\mu_I)$ and $\langle G^2\rangle(\mu_I)$ can be obtained by Eqs. (10) and (11) from the effective pressure

$$P_{\text{eff}}(\mu_I) = -\varepsilon_{\text{vac}} + P_\pi(\mu_I). \tag{12}$$

Here, $\varepsilon_{\text{vac}} = \frac{1}{4}\langle\theta_{\mu\mu}\rangle$ is the nonperturbative vacuum energy density at $\mu_I = 0$, where

$$\langle\theta_{\mu\mu}\rangle = -\frac{b}{32\pi^2}\langle G^2\rangle + \sum_{q=u,d}m_q\langle\bar{q}q\rangle \tag{13}$$

is the trace of the energy–momentum tensor, and $P_\pi(\mu_I)$ is the pressure created by pions at a finite isospin chemical potential. The quark and gluon condensates are given by the formulas

$$\langle\bar{q}q\rangle(\mu_I) = -\frac{\partial P_{\text{eff}}}{\partial m_q}, \tag{14}$$

$$\langle G^2\rangle(\mu_I) = \hat{D}P_{\text{eff}}, \tag{15}$$

where the operator $\hat{D}$ is introduced in Eq. (10) in the form

$$\hat{D} = \frac{32\pi^2}{b}\left(4 - \mu_I\frac{\partial}{\partial\mu_I} - \sum_q m_q\frac{\partial}{\partial m_q}\right). \tag{16}$$

Below, the case $\mu_I = 0$ is considered with the use of the low-energy theorem for the derivative of the gluon condensate with respect to the quark mass [15]

$$\begin{aligned}\frac{\partial}{\partial m_q}\langle G^2\rangle &= \int d^4x\langle G^2(0)\bar{q}q(x)\rangle\\ &= -\frac{96\pi^2}{b}\langle\bar{q}q\rangle + O(m_q),\end{aligned} \tag{17}$$

where $O(m_q)$ stands for terms linear in the masses of light quarks. The resulting relation has the form

$$\begin{aligned}\frac{\partial\varepsilon_{\text{vac}}}{\partial m_q} &= -\frac{b}{128\pi^2}\frac{\partial}{\partial m_q}\langle G^2\rangle + \frac{1}{4}\langle\bar{q}q\rangle\\ &= \frac{3}{4}\langle\bar{q}q\rangle + \frac{1}{4}\langle\bar{q}q\rangle = \langle\bar{q}q\rangle.\end{aligned} \tag{18}$$

It is noteworthy that three-fourths of the quark condensate originates from the gluon part of the nonperturbative energy density of vacuum. The following expression for the gluon condensate is obtained similarly:

$$-\hat{D}\varepsilon_{\rm vac} = \langle G^2\rangle. \tag{19}$$

In order to obtain the dependence of the quark and gluon condensates on the isospin chemical potential μ_I, it is convenient to use the Gell-Mann–Oakes–Renner relation

$$f_\pi^2 m_\pi^2 = -\frac{1}{2}(m_u + m_d)\langle \bar{u}u + \bar{d}d\rangle = (m_u + m_d)\Sigma, \tag{20}$$

where $\Sigma = |\langle \bar{u}u\rangle| = |\langle \bar{d}d\rangle|$. Then, the following relations are derived:

$$\frac{\partial}{\partial m_q} = \frac{\Sigma}{f_\pi^2}\frac{\partial}{\partial m_\pi^2}, \tag{21}$$

$$\sum_q m_q \frac{\partial}{\partial m_q} = (m_u + m_d)\frac{\Sigma}{f_\pi^2}\frac{\partial}{\partial m_\pi^2} = m_\pi^2\frac{\partial}{\partial m_\pi^2}, \tag{22}$$

$$\hat{D} = \frac{32\pi^2}{b}\left(4 - \mu_I\frac{\partial}{\partial \mu_I} - m_\pi^2\frac{\partial}{\partial m_\pi^2}\right). \tag{23}$$

In the tree level of the chiral perturbation theory, the pressure from which it is possible to obtain the quark and gluon condensates in the pion condensate phase ($\mu_I > m_\pi$) has the form [10]

$$P_\pi^{\rm ChPT} = \frac{1}{2}f_\pi^2\mu_I^2\left(1 - \frac{m_\pi^2}{\mu_I^2}\right)^2. \tag{24}$$

According to Eqs. (12), (14), (21), and (24), the quark condensate at $\mu_I > m_\pi$ is given by the expression

$$\Sigma(\mu_I) = \Sigma\frac{m_\pi^2}{\mu_I^2}. \tag{25}$$

Similarly, according to Eqs. (12), (15), (23), and (24), the gluon condensate in the pion condensate phase has the form

$$\begin{aligned}\langle G^2\rangle(\mu_I) &= \langle G^2\rangle \\ &+ \frac{32\pi^2}{b}f_\pi^2\mu_I^2\left(1 - 3\frac{m_\pi^2}{\mu_I^2} + 2\frac{m_\pi^4}{\mu_I^4}\right).\end{aligned} \tag{26}$$

At $\mu_I < m_\pi$, the system is in the vacuum phase and the condensates are equal to their vacuum values. It is seen that the quark condensate decreases with increasing μ_I as $\frac{m_\pi^2}{\mu_I^2}$. This behavior is in good agreement with early numerical calculations [8, 9] and with recent lattice QCD calculations [37] at zero temperature.

The gluon condensate in the region of the pion condensate ($\mu_I > m_\pi$) first decreases, reaches its minimum value at $\mu_I = 2^{1/4}m_\pi \simeq 1.19m_\pi$, and then increases. At the point $\mu_I = 2^{1/2}m_\pi \simeq 1.41m_\pi$, it becomes equal to its vacuum value $\langle G^2\rangle$ and then continues to increase with μ_I. This phenomenon can be called gluon catalysis at a finite isospin chemical potential.

4. To summarize, the nonperturbative QCD vacuum at a finite isospin chemical potential has been studied. Low-energy relations for the quark and gluon condensates have been derived ab initio. Analytical expressions for the quark and gluon condensates in the pion condensate phase have been obtained in the tree level of the chiral perturbation theory. It has been shown that the quark condensate decreases with increasing μ_I, whereas the gluon condensate increases.

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