

AMERICAN JOURNAL of PHYSICS

Devoted to the Instructional and Cultural Aspects of Physical Science

91410

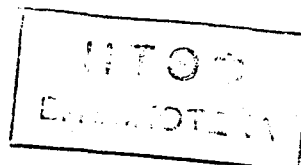
ФИЗИКА № 7-10
(американ журнал оф физике)
По подписке 1968

VOLUME 36

NUMBER 10 v

OCTOBER 1968

11610



Published monthly for the
AMERICAN ASSOCIATION OF PHYSICS TEACHERS
BY THE
AMERICAN INSTITUTE OF PHYSICS

PRINCE & LEMON STREETS, LANCASTER, PA., or 335 EAST 45 STREET, NEW YORK, N. Y. 10017

The Advantage of Teaching Relativity with Four-Vectors.

ROBERT W. BREHME

Wake Forest University, Winston Salem, North Carolina 27109

(Received 11 April 1968; revision received 27 May 1968)

An argument is given that teaching relativity from the beginning using four-vectors provides a powerful and correct way of thinking that also conforms to everyday methods of making measurements. In particular, it is pointed out that the emphasis on the relativistic mass is both undeserved and misleading.

The theory of relativity has its clearest and simplest expression in the symbolism of four-vectors. This article is written to illustrate some of the uses to which the four-vector notation can be applied with ease to a few typical elementary problems in relativity. If a justification for such a pedagogical approach need be given, it is that all too often the simplest problems in relativity are made unnecessarily difficult by the cumbersome use of non-four-vectors.

A four-vector is any quantity of four *components* which transforms under a Lorentz transformation like the four components of the space-time infinitesimals, dx , dy , dz , and cdt . For the situation in which the spatial axes of the two frames, connected by the Lorentz transformation, are parallel, and the motion occurs along their common x axes, the transformation is

$$dx' = dx \sec \alpha + cdt \tan \alpha, \quad (0.1a)$$

$$dy' = dy, \quad (0.1b)$$

$$dz' = dz, \quad (0.1c)$$

$$cdt' = cdt \sec \alpha + dx \tan \alpha, \quad (0.1d)$$

where the angle α is defined by $V_x/c = \sin \alpha$, and the relative velocity between the frames is V_x . The corresponding transformation for any four-vector, F , is

$$F_x' = F_x \sec \alpha + F_t \tan \alpha, \quad (0.2a)$$

$$F_y' = F_y, \quad (0.2b)$$

$$F_z' = F_z, \quad (0.2c)$$

$$F_t' = F_t \sec \alpha + F_x \tan \alpha. \quad (0.2d)$$

The inverse transformation is obtained by exchanging primes and unprimes and changing the sign of α .

The *invariant square*, F^2 , is defined

$$F^2 = \pm (F_x^2 + F_y^2 + F_z^2 - F_t^2), \quad (0.3)$$

where the sign is chosen to make F^2 positive. This quantity has the same value relative to all frames connected by the Lorentz transformation.

I. THE PROPER VELOCITY

As the simplest example of the superiority of the four-vector over the conventional non-four-vector, we compare the four-vector of *proper velocity* with the three-vector *coordinate velocity*. The four components of the proper velocity are defined to be $u_x = dx/d\tau$, $u_y = dy/d\tau$, $u_z = dz/d\tau$ and $u_t = cdt/d\tau$, where $d\tau$ is the invariant time measured by a clock carried with the object during which time the object travels the component distances dx , dy , dz , measured relative to an inertial frame. The value of dt is determined by *two* clocks, at rest in and synchronized relative to the frame, one placed at the spatial point from which the beginning of the spatial interval dx , dy , dz is measured, the other placed at the end of this interval. Then dt is the *difference* in the reading of the first clock at the moment the object passes and the reading of the second clock at the moment the object passes.

The three components of the coordinate velocity are defined to be $v_x = dx/dt$, $v_y = dy/dt$, and $v_z = dz/dt$. If a fourth component were to be defined in analogy to the proper velocity, it would be $v_t = cdt/dt = c$. The now *four* components of coordinate velocity clearly do not constitute a four-vector since they do not transform under the Lorentz transformation in the right way. The transformation law for the spatial components of the coordinate velocity, known as the *Einstein* (or *relativistic*) velocity addition theorem, is awkward and difficult to use in any but the very simplest situations.

Of the two velocities, the *proper* velocity is that which is encountered in our everyday experience. The automobile driver who wishes to test the accuracy of his speedometer does so most likely by measuring, *with a clock he carries with him*, the time during which he traverses a distance, measured by, say, mile markers placed along the highway. He calculates his average velocity to be the ratio of the distance measured *relative to the earth* to the time measured *in the car*. This velocity is by definition the *proper* velocity. To measure his coordinate velocity, he would need *two* synchronized (relative to the earth) clocks at rest (relative to the earth) at the first and second mile markers. To find the time interval, he would subtract the reading of the first clock from the second.

The relationship between the coordinate velocity and the proper velocity is found first by noting that

$$c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2). \quad (1.1)$$

Hence

$$c^2 = u_t^2 - (u_x^2 + u_y^2 + u_z^2), \quad (1.2)$$

and

$$(d\tau/dt)^2 = 1 - v^2/c^2, \quad (1.3a)$$

or

$$d\tau = dt(1 - v^2/c^2)^{1/2}. \quad (1.3b)$$

Thus

$$u_x = dx/d\tau = (dx/dt)(dt/d\tau),$$

or

$$u_x = v_x / (1 - v^2/c^2)^{1/2}, \quad (1.4a)$$

$$u_y = v_y / (1 - v^2/c^2)^{1/2}, \quad (1.4b)$$

$$u_z = v_z / (1 - v^2/c^2)^{1/2}, \quad (1.4c)$$

$$u_t = c / (1 - v^2/c^2)^{1/2}. \quad (1.4d)$$

The coordinate velocity of light is c , but its proper velocity is *infinite*! There is no speed limit attached to the *proper* velocity. For low speeds the four components of the proper velocity nearly equal the corresponding components of the coordinate velocity. The spatial components of the proper velocity of an object at rest are zero, but the time component is the speed of light.

If we measure the speed between two frames of reference with the proper velocity rather than with the coordinate velocity, we have $U_x/c = \tan\alpha$ and $U_t/c = \sec\alpha$, where, as before $V/c = \sin\alpha$. The

Lorentz transformation now appears

$$dx' = (U_t/c)dx + (U_x/c)c dt, \quad (1.5a)$$

$$cdt' = (U_t/c)c dt + (U_x/c)dx. \quad (1.5b)$$

The first equation appears more like the Galilean transformation than does the Lorentz transformation expressed in the usual way. In particular, the awkward square root is absent.

II. THE PROPER ACCELERATION

Four varieties of acceleration are possible. For the x component we have our choice of dv_x/dt , du_x/dt , $dv_x/d\tau$, and $du_x/d\tau$. The first is called the *coordinate acceleration* and the last the *proper acceleration*. The second and third are hybrids and have no names. The first makes use of at least *three* coordinate clocks and is clumsy to use in practice. The last needs only the clock carried with the object. Because the proper time is an invariant, the *proper acceleration transforms like a four-vector*. The transformation properties of the coordinate acceleration are so complicated that they are rarely written down, and once seen they are difficult to interpret. The hybrids neither are four vectors nor have simple transformation laws.

Differentiating Eq. (1.2) with respect to the proper time, we have

$$0 = u_t a_t - (u_x a_x + u_y a_y + u_z a_z), \quad (2.1)$$

where $a_x = du_x/d\tau$, etc. Since the proper acceleration is a four-vector, its components form an invariant square:

$$a^2 = a_x^2 + a_y^2 + a_z^2 - a_t^2. \quad (2.2)$$

In the momentary rest frame of the object, $u_x = u_y = u_z = 0$ and $u_t = c$. Hence from Eq. (2.1), $a_t = 0$. Therefore, according to Eq. (2.2), the *invariant* a is the acceleration of the object in its momentary rest frame. It is called the *local* acceleration.

Let the primed frame be the momentary rest frame of the object and the unprimed frame the frame in which the object moves along the x axis at the proper velocity u_x . According to Eq. (0.2), the Lorentz transformation for the acceleration can be written

$$a_x = a_x' \sec\alpha + a_t' \tan\alpha, \quad (2.3a)$$

$$a_t = a_t' \sec\alpha + a_x' \tan\alpha. \quad (2.3b)$$

But $a_x' = a$, $a_t' = 0$, and $u_x = c \tan \alpha$. Thus Eq. (2.3a) can be written

$$du_x/d\tau = c \sec^2 \alpha (d\alpha/d\tau) = a \sec \alpha. \quad (2.4)$$

Integrating, we have

$$c \ln(\sec \alpha + \tan \alpha) = \int a d\tau, \quad (2.5a)$$

or

$$u_x + u_t = c \exp\left(\int \frac{a d\tau}{c}\right). \quad (2.5b)$$

Since $u_t^2 - u_x^2 = c^2$, we have

$$\begin{aligned} \frac{u_x}{c} &= \frac{1}{2} \left[\exp\left(\int \frac{a d\tau}{c}\right) \exp\left(-\int \frac{a d\tau}{c}\right) \right] \\ &= \sinh\left(\int \frac{a d\tau}{c}\right), \end{aligned} \quad (2.6a)$$

$$\begin{aligned} \frac{u_t}{c} &= \frac{1}{2} \left[\exp\left(\int \frac{a d\tau}{c}\right) + \exp\left(-\int \frac{a d\tau}{c}\right) \right] \\ &= \cosh\left(\int \frac{a d\tau}{c}\right). \end{aligned} \quad (2.6b)$$

These results could have been obtained immediately from the substitution of $\sinh \theta$ for $\tan \alpha$. It is a matter of personal preference at which point the hyperbolic functions are explicitly introduced. If the local acceleration is constant, then $\int a d\tau = a\tau$, and

$$a_x = a \sec \alpha = a \cosh(a\tau/c), \quad (2.7)$$

$$u_x = c \tan \alpha = c \sinh(a\tau/c), \quad (2.8a)$$

$$u_t = c \sec \alpha = c \cosh(a\tau/c), \quad (2.8b)$$

$$\begin{aligned} x &= \int u_x d\tau = c \int \sinh\left(\frac{a\tau}{c}\right) d\tau \\ &= (c^2/a) [\cosh(a\tau/c) - 1], \end{aligned} \quad (2.9a)$$

$$\begin{aligned} ct &= \int u_t d\tau = c \int \cosh\left(\frac{a\tau}{c}\right) d\tau \\ &= (c^2/a) \sinh(a\tau/c). \end{aligned} \quad (2.9b)$$

The constants of integration are chosen so that

$x=0, t=0$ at $\tau=0$. Since

$$\cosh^2(a\tau/c) - \sinh^2(a\tau/c) = 1,$$

we have

$$(1 - ax/c^2)^2 - (at/c)^2 = 1. \quad (2.10)$$

The equation describes a rectangular hyperbola. To arrive at Eq. (2.10) via the coordinate acceleration and the requirement that d^2x/dt^2 be a constant in the momentary rest frame, is long and difficult.

III. DYNAMICS AND NEWTON'S SECOND LAW

The usual approach to dynamics in relativity is through the Lewis-Tolman collision. No appeal is made to Newton's first or second laws, probably because, when written using three-vectors, the relativistically correct equations are not suggested by any extrapolation of the classical expressions. However, once we become accustomed to thinking entirely in terms of four-vectors, the correct expression for Newton's second law is the obvious and trivial extrapolation of the classical rule.

We introduce an invariant scalar, m , the mass of the object. We define the components of the proper force acting on the object to be

$$F_x = ma_x, \quad (3.1a)$$

$$F_y = ma_y, \quad (3.1b)$$

$$F_z = ma_z. \quad (3.1c)$$

Since the acceleration is a four-vector and the mass a scalar invariant, F_x , F_y , and F_z are the spatial components of a four-vector, the fourth component of which must be

$$F_t = ma_t. \quad (3.1d)$$

Since the force so defined is a four-vector, it transforms under a Lorentz transformation according to Eq. (0.2).

As an example, the proper force producing a constant local acceleration is constant in the rest frame of the object. Thus, if the "object" is a rocket ship, a constant thrust from the rocket motors (measured in the ship frame) produces a constant local acceleration, and the equations of motion of the ship are given by Eqs. (2.7)-(2.10).

IV. MOMENTUM AND ENERGY

The relativistic definition of *proper momentum* clearly ought to be

$$p_x = mu_x, \quad (4.1a)$$

$$p_y = mu_y, \quad (4.1b)$$

$$p_z = mu_z, \quad (4.1c)$$

$$p_t = mu_t. \quad (4.1d)$$

Since mass is invariant, this definition guarantees that momentum is a four-vector. When m is constant, we can write $F_x = dp_x/d\tau$, etc., so that the proper force is the proper time-rate-of-change of proper momentum. If m is not constant, then the force so defined is not the product of mass and acceleration, but then neither is it this product in classical mechanics under this circumstance. The proper force remains a four-vector, however.

Newton's second law is in fact merely a definition of force in both classical and relativistic mechanics. The law governing *interactions* is the third law, the *law of momentum conservation*.

Because the Lorentz transformation is linear, the sum of four-vectors is also a four-vector. The total momentum of a system of particles is therefore a four-vector, and changes in this total momentum are also four-vectors. If the components of the total proper momentum are conserved in the unprimed frame, then they are conserved in the arbitrary primed frame. In other words, if the conservation of momentum is to apply simultaneously to all frames connected by the Lorentz transformation, then momentum must be a four-vector. It is for this reason that momentum cannot be defined as mv , where m is a scalar invariant and v the coordinate velocity. Momentum so defined could be conserved in only one frame at a time—whereas, if we are to avoid the philosophical problem of a preferred frame, the conservation law ought to apply independently of the frame of reference.

The fourth component of the momentum is $p_t = mc(1 + u^2/c^2)^{1/2}$, where $u^2 = u_x^2 + u_y^2 + u_z^2$. Expanding, we have

$$cp_t = mc^2 + \frac{1}{2}mu^2 + \text{etc.} \quad (4.2)$$

Since $\frac{1}{2}mu^2$ is the classical kinetic energy, we define

for the relativistic case the *kinetic energy*

$$\mathcal{E}_K = cp_t - mc^2, \quad (4.3a)$$

the *rest energy* to be

$$\mathcal{E}_0 = mc^2, \quad (4.4)$$

and the *total energy* to be

$$\mathcal{E} = cp_t. \quad (4.4)$$

Thus, Eq. (4.3a) is

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_K \quad (4.3b)$$

For constant m , $F_t = dp_t/d\tau = d\mathcal{E}/cd\tau$. Thus F_t is the *proper* time rate of change of the total energy (divided by c).

Next, consider Eq. (2.1) multiplied by the mass of the object. Then

$$0 = F_t u_t - (F_x u_x + F_y u_y + F_z u_z), \quad (4.4a)$$

or

$$d\mathcal{E} = F_t cd\tau = (F_x dx + F_y dy + F_z dz) (c/u_t) \quad (4.4b)$$

The term on the left is the change in the total energy of the object. We define the right side of the equation as the *work* of the force on the object.

V. THE BANE OF THE RELATIVISTIC MASS

The historical development of a physical theory does not necessarily guarantee the best philosophical viewpoint. The concept of the *relativistic mass*, which has played such a prominent role in the description of relativity, is one that deserves a much more subordinate status than it enjoys, particularly since its interpretation has been a source of confusion to many teachers of relativity.

The concept arose because momentum, defined as mv , cannot be simultaneously conserved in all Lorentz frames as long as mass is an invariant. We have seen above that the correct form for momentum is (for the x direction) $p_x = mu_x$. Since $u_x = v_x/(1 - v^2/c^2)^{1/2}$, we may write

$$p_x = mv_x/(1 - v^2/c^2)^{1/2}. \quad (5.1a)$$

Similarly,

$$p_t = mc/(1 - v^2/c^2)^{1/2}. \quad (5.1b)$$

Now the early relativists, deriving Eq. (5.1) via the Lewis-Tolman collision, sought to preserve the

usefulness of the coordinate velocity and were thereby led to associate the square root with the mass, m . They defined the *relativistic mass* \tilde{m} to be $\tilde{m} = m/(1 - v^2/c^2)^{1/2}$. Thus, $p_x = \tilde{m}v_x$ and $p_t = \tilde{m}c$. While the equations are of course correct, their form is misleading for the following reasons.¹

1. Four-vectors arise in mechanics solely because of the transformation properties of the coordinate infinitesimals—that is, because of the *kinematic* properties of space time. The dynamical behavior of motion is “relativistic” only through the relativistic nature of kinematics. Dynamical four-vectors appear only from the multiplication of the dynamical invariant, mass, and a kinematic four-vector. By assigning mass a relativistic character, we obscure both the simplicity and the essentially kinematic nature of relativity.

2. Unless one is aware of the proper velocity and its relationship to the coordinate velocity, it is not immediately evident that $\tilde{m}v_x$ and $\tilde{m}c$ are the x and t components of a four-vector. With the use of four-vectors, it is immediately evident that \tilde{m} is proportional to the time component of the momentum four-vector.

3. The equation for the time component of momentum, $cp_t = \mathcal{E} = \tilde{m}c^2$, while perhaps the best known of all equations in physics, has also been the most frequently misinterpreted. The equation says that relativistic mass is total energy (divided by c^2). The equation does *not* say that relativistic mass can be *converted* into energy and that if a mass \tilde{m} disappears an energy, $\mathcal{E} = \tilde{m}c^2$ appears in its place. If \tilde{m} disappears so also does \mathcal{E} . When no external forces act on a system so that the four components of the total momentum are conserved, then the total energy remains constant and so also does the relativistic mass.

4. While the relativistic mass is identically proportional to the total energy, it is necessary to regard mass, in a certain sense, as a *form* of energy. The internal energy of a body, whether chemical, potential, or thermal, etc., contributes to its mass. But because this energy may leave the body in a form which has zero mass, we cannot equate energy identically with this mass. It is the confusion between the properties of mass and relativistic mass which argues for a de-emphasis of

the relativistic mass. To see how the concept of mass as a form of energy arises, we first define a *structureless* particle to be one for which the mass is constant during the existence of the particle. We next regard an *object* to be an aggregate of these particles, moving about at random within the boundaries of the object. Relative to any frame, the components of the momentum, P_x , etc., of the object are defined to be the sum of the corresponding components of the momenta of the particles, p_x , etc. That is, $P_x = \sum p_x$, etc. The mass of the object, M , is then defined by

$$(Mc)^2 = P_t^2 - (P_x^2 + P_y^2 + P_z^2), \quad (5.2)$$

an equation analogous to that satisfied by the mass, m , of any of the particles,

$$(mc)^2 = p_t^2 - (p_x^2 + p_y^2 + p_z^2). \quad (5.3)$$

The rest frame of the object is that frame for which $P_x = P_y = P_z = 0$. In this frame, $M = P_t/c$. Because of the random motion of the particles, they are not all at rest relative to the rest frame, and therefore each may contribute some kinetic as well as rest energy to the total energy. Thus

$$\begin{aligned} cP_t &= \sum (\mathcal{E}_0 + \mathcal{E}_k) \\ &= \sum (mc^2 + \mathcal{E}_k), \end{aligned}$$

and

$$M = \sum (m + \mathcal{E}_k/c^2). \quad (5.4)$$

The mass of the object exceeds the total mass of the particles by an amount proportional to the kinetic energy of the particles, measured relative to the rest frame of the object.

Suppose the object cools by thermal radiation and the kinetic energy of random motion decreases. Although the total mass of the particles in the object remains constant, the mass of the object itself decreases. The loss in mass, ΔM , is exactly compensated for by the energy of radiation, \mathcal{E}_r , according to $\mathcal{E}_r = c^2 \Delta M$. Because this radiant energy propagates at the speed of light, its mass is zero. That is, for all things moving at the speed of light,

$$0 = u_t^2 - (u_x^2 + u_y^2 + u_z^2),$$

and

$$mc^2 = 0 = p_t^2 - (p_x^2 + p_y^2 + p_z^2). \quad (5.5)$$

¹The word *mass* appearing without the adjective *relativistic*, means the invariant mass introduced in Sec. III.

Hence, the mass of the object has decreased without the appearance of this mass in any other object. However, relativistic mass, being proportional to the energy, is still conserved. The relativistic mass of radiation, \tilde{m}_r , is given by $\tilde{m}_r = \mathcal{E}_r/c^2$, and it is also equal to the loss in mass, ΔM , when the radiation is observed from the rest frame of the object. Nevertheless, in this case, it is inconsistent to think of mass as being ejected from the object,

since radiant energy has zero mass. Mass in this example is *converted* into radiant energy.

In general, any time the internal energy of an object appears as radiant energy (e.g., radiant cooling or positron annihilation), or does mechanical work by increasing the kinetic energy of the object or some other object (e.g., a heat engine), the mass of the object is converted into some other form of energy.

AMERICAN JOURNAL OF PHYSICS VOLUME 36, NUMBER 10 OCTOBER 1968

Lorentz Transformation*

A. L. HARVEY

Queens College of The City University of New York, Flushing, New York 11367

(Received 17 April 1968)

A simple-straightforward derivation of the most general form of the homogeneous Lorentz transformation is presented. In the process, the well-known decomposition of a generic element of the group into two pure rotations and a "one-dimensional" Lorentz transformation is obtained in a physically obvious manner.

Although derivations of the Lorentz transformation abound, most of them leave something to be desired from the point of view of the average advanced undergraduate physics major. The derivations, if simple, are not general; if they are general, they are perhaps too abstract.¹ The purpose of this article is to provide a simple-

straightforward derivation of the most general form of the (homogeneous) Lorentz transformation. Though some of the details are familiar, there are a number of innovations of approach resulting in a derivation at once more transparent, completely general, and possessed of unique results.

* Supported in part by a grant from the U. S. Office of Naval Research.

¹ The basic reference here is A. Einstein, *Ann. Physik* 17, 891 (1905). The derivation is "one-dimensional," purely operational in concept, and quite sophisticated. The typical simple derivation is exemplified by H. Goldstein, *Classical Mechanics* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1950). More general treatments based primarily on space-time concepts and the invariance of the speed of light are given by J. Aharoni, *The Special Theory of Relativity* (Oxford University Press, London, 1959); C. Møller, *The Theory of Relativity* (Oxford University Press, London, 1952); or H. M. Schwartz, *Introduction to Special Relativity* (McGraw-Hill Book Co., New York, 1968). Unique advanced geometric treatments are given by J. L. Synge, *Relativity: The Special Theory* (North-Holland Publ. Co., Amsterdam, 1964) and A. D. Fokker, *Time and Space, Weight and Inertia* (Pergamon Press Ltd., London, 1965). Of historical interest are G. Herglotz, *Ann. Physik*, 36, 497 (1911) and L. Silberstein, *The Theory of Relativity* (MacMillan and Company Ltd., London, 1914).

The concern here is with the details of the actual derivation as distinguished from the conceptual basis for the initial assumptions. Thorough discussions of the latter exist in the literature.² However, it is necessary as a minimum to state the assumptions used here. They are:

- (1) There exists a set of preferred inertial observers.
- (2) The speed of light is the same for each observer of the set and is independent of the motion of the source.
- (3) The (three-) space of each observer is flat.

² See, e.g., for elementary or pedagogical discussions, E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (W. H. Freeman and Co., San Francisco, Calif., 1963); H. M. Schwartz, *Introduction to Special Relativity* (McGraw-Hill Book Co., New York, 1968); or W. G. V. Rosser, *An Introduction to the Theory of Relativity* (Butterworths Scientific Publishers Ltd., London, 1964).